Types of Surjection

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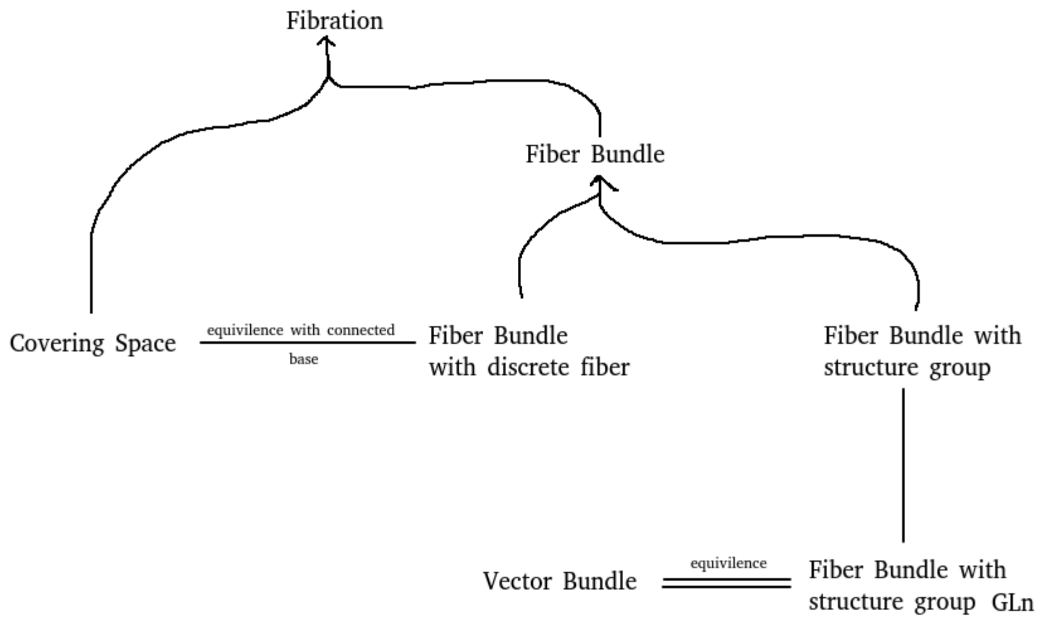
1 Comparison



The arrow indicates that the thing is an example or special case of the thing above it. Conceptually perhaps we can put it like this: You start with vector bundles, if you remove the condition that the fibers are vector spaces you just get a fiber bundle, if you want to consider everything only up to homotopy you get fibrations and if you want to replace $\mathrm{GL}_n(\mathbb{R})$ with a different group you get fiber bundles with different structure groups.

2 Vector Bundles

This section is from [MS16]. Consider a topological space B. A *real vector bundle* ξ is a surjection of topological spaces

$$\pi : E \to B,$$

along with the structure of a real vector space on each fiber $\pi^{-1}(b)$ that satisfies the local triviality condition: for all $b \in B$ there is some neighbourhood U some $n \in \mathbb{N}$ and some homeomorphism

$$h : U \times R^n \to \pi^{-1}(U)$$

that is fiberwise a linear isomorphism. We call E the total space and B the base space. Note that here the dimension of the fiber can depend on the point. A bundle map or morphism is a continuous map on the total spaces

$$g : E \to E'$$

such that $E_b \mapsto E_{b'}$ isomorphically (as topological vector spaces) for some $b, b' \in B$. This condition allows g to descend to a map on B by defining $\bar{g}(b) \ := \ b'$. A bundle is *trivial* if it is isomorphic to $B \times \mathbb{R}^n$.

3 Fiber Bundle

A fiber bundle is a locally trivial space, where the fibers dont need to be vector spaces. A fiber bundle is a surjection $\pi : E \to B$ and a fiber F satisfying the local triviality condition that around every point there is a neighbourhood $x \in U \subseteq B$ and a homeomorphism

$$\varphi : \pi^{-1}(U) \to U \times F$$

such that the diagram commutes

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"] & U \times F \arrow[dl, "proj_1"] \\ U & \end{tikzcd}$$

Remark: This definition implies that the fiber over a point is always homeomorphic to F.

4 Bundles with Groups actions

Throughout we will consider all bundles to be fiber bundles, so all conditions will be ontop of that. Actually there are many ways to talk about G actions on bundles etc and they are often only equivilent in the case of certain smooth manifolds if at all.
`https://math.stackexchange.com/questions/650145/difference-between-various-type-of-bundles-having-a-g`
`https://math.stackexchange.com/questions/399035/equivalence-of-definitions-of-principal-g-bundle`

We will follow [Ste16], although there he defines these as just "fiber bundles" we will call them fiber bundles with structure group.

Given a topological group and space G, Y respectively we say that G is a transformation group for Y relative to $\eta : G \times Y \to Y$ if

- η is continuous
- $\eta(id, -) = id_Y$
- (Associativity) $\eta(gh, y) = \eta(g, \eta(h, y))$

This defines a homomorphism (the right is a group under composition)

$$G \to \mathrm{Aut}(Y) \subseteq \mathrm{Hom}(Y, Y)$$
$$g \mapsto (y \mapsto gy)$$

note that left multiplication is obviously a homeomorphism because one has the inverse given by multiplying by the inverse group element. G is *effective* if $gy = y$ for all y implies that $g = id$. If G is effective then the map above is *injective*. In particular G can be though of as a subgroup of $\mathrm{Aut}(Y)$.

All actions are assumed to be effective. A coordinate bundle, or fiber bundle with structure group G is then a fiber bundle $(F, \pi : E \to B)$ with an effective group transformation of G on F that satisfies the following compatability conditions. If we have a system of local trivialisations $\varphi_j : V_j \times F \to \pi^{-1}(V_j)$ that cover B then for every i, j and $x \in V_i \cap V_j$

$$\varphi_j(x, -)^{-1} \circ \varphi_i(x, -) : F \to F$$

is a homeomorphism **we require that it is given by the action of a group element** $g_{ij} \in G$, and that **the assignment**

$$V_i \cap V_j \to G$$
$$x \mapsto \varphi_j(x, -)^{-1} \circ \varphi_i(x, -)$$

is continuous.

Remark: Strictly speaking we should take equivilence classes of such structures where we identify bundles which have "compatible" systems of local trivilisations. This is a standard detail that is unenlightening.

A morphism of a fiber bundles with structure groups from $(F, \pi : E \to B, G)$ to $(F, \pi : E' \to B', G)$, note that they have the same fiber and structure group, is a map of the total spaces

$$h : E \to E'$$

satisfying the compatibility conditions. First h is a fiberwise homeomorphism. h defines a map on the base space $\bar{h}$, that is if h sends $F_x \mapsto F'_{x'}$ then $\bar{h}(x) = x'$. Then if we consider $x \in V_j \cap \bar{h}^{-1}(V'_k)$

$$\varphi'_k(x', -)^{-1} h \varphi_j(x, -) : F \to F_x \to F'_{x'} \to F$$

it is the composition of homeomorphisms and therefore a homeomorphism **we require that it is given by the action of a group element**, say $g_{kj}(x)$, and that the map

$$V_j \cap \bar{h}^{-1}(V'_k) \to G$$

$$x \mapsto g_{jk}(x)$$

is continuous.

Remark: Any fiber bundle has an a priori structure group given by $\mathrm{Aut}(F)$. Subgroups of this group will *not necissarily* be *representable*, that is the local trivialisations chosen will determine which subgroups are also suitable structure groups. We might have an action of the full group, which would induce an action of the subgroup, what remains is whether or not the subgroup represents all of the local trivialisations. Steenrod calls this a "reduction of the structure group".

4.1 Examples

4.1.1 Vector Bundles

Theorem. *There is an equivalence of categories*

$$\mathbb{R}^n - \textit{bundles} \leftrightarrow \textit{fiber bundles with structure group } \mathrm{GL}_n(\mathbb{R})$$

From a vector bundle $\pi : E \to B$ with a local trivialization (φ, U) we get that φ is fiberwise a linear isomorphism and so in particular $\varphi_i^{-1}\varphi_j$ is a linear isomorphism of $\mathbb{R}^n$ and hence represented by an element of $\mathrm{GL}_n(\mathbb{R})$. Thus not only does $\mathrm{GL}_n(\mathbb{R})$ act as a subgroup of $\mathrm{Aut}(\mathbb{R}^n)$ but all the local trivialisations are representable by it.

In reverse a fiber bundle with structure group $\mathrm{GL}_n(\mathbb{R})$ requires that the trivilisations are given by linear isomorphism, from which we can transport a vector space structure to all the fibers.

The correspondence of morphisms is similarly clear. Namely being represented by an element of $\mathrm{GL}_n(\mathbb{R})$ in the structure group language coresponds exactly to the maps being linear isomorphisms in the vector bundle language.

4.1.2 Vector Bundles with Metric

A metric gives a reduction of the structure group from $\mathrm{GL}_n(\mathbb{R})$ to $\mathrm{O}(\mathbb{R})$. That is the local trivilisations are now representable by $\mathrm{O}_n(\mathbb{R})$ matricies as they are required to not just preserve the linear structure but also the inner product.

4.1.3 Oriented Vector Bundles

This is similarly a reduction of the structure group to $\mathrm{GL}_n^+(\mathbb{R})$, requiring to preserve orientation is the same as having the transition matrix be positive definite.

5 Fibrations

A fibration is a surjection that satisfies a homotopy lifting property. If we have a map of topological spaces $p : E \to B$ then it is said to satisfy the homotopy lifting property with respect to a space X if for every diagram we have a lift

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\tilde{h}_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "h"] \arrow[ur, dashed, "\exists"] & B \end{tikzcd}$$

A *Hurewicz fibration* is a map that satisfies the homotopy lifting property for all spaces X. A Serre fibration is a map satisfying the homotopy lifting property for all CW complexes X.

Remark: In this case the fiber over a point is unique *only up to homotopy.*

6 Covering Spaces

A covering space is a surjective map $\pi : E \to B$ such that around every point there is a neighbourhood $x \in U \subseteq B$ and a discrete space D such that

$$\pi^{-1}(U) \cong \sqcup_{d \in D} U$$

Remark: It is clear that the fibers in this case are discrete.

References

[MS16] John Willard Milnor and James D. Stasheff. *Characteristic Classes: AM-76*. Number 76 in Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2016.

[Ste16] Norman Earl Steenrod. *The Topology of Fibre Bundles. (PMS-14)*. Number 14 in Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 2016.